MATH 3235 Probability Theory

10/20/2022.

$X \qquad f_X(x)$

$g(X) = Y \qquad$ distribution of Y?

$$\mathbb{P}(Y \in A) = \int_A f_Y(y)\,dy$$

$$\parallel$$

$$\mathbb{P}(X \in g^{-1}(A)) = \int_{g^{-1}(A)} f_X(x)\,dx =$$

$y = g(x)$

$$= \int_A f_X(g^{-1}(y)) \frac{dx}{dy}\,dy$$

$$= \int_A f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)\,dy$$

For every A you have

$$\int_A F(x)\,dx = \int_A G(x)\,dx \implies$$

$$F(x) = G(x) \quad \forall x$$

$(X, Y) \qquad f_{X,Y}(x,y)$

$(U, V) = (u(X,Y), v(X,Y))$

Assume that we can write

$$X = x(U, V)$$
$$Y = y(U, V)$$

$$f_{U,V}(u,v) = f_{X,Y}(x(u,v), y(u,v)) \, |J(u,v)|$$

$$J(u,v) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Jacobian.

A 2×2 positive definite

$\vec{x} \in \mathbb{R}^2$

$$(\vec{x}, A\vec{x}) > 0$$

$\vec{x} \neq 0$

$$a_{11} x_1^2 + (a_{12} + a_{21}) x_1 x_2 + a_{22} x_2^2 > 0$$

$$\forall x_1, x_2 \neq 0$$

$a_{12} = a_{21}$

$$\vec{x} = (x_1, x_2)$$

$$\frac{1}{Z} e^{-\frac{(\vec{x}, A\vec{x})}{2}} = f_{X_1, X_2}(x_1, x_2)$$

$$Z = \int_{\mathbb{R}^2} e^{-\frac{(\vec{x}, A\vec{x})}{2}} dx_1\, dx_2$$

$A = Id$

$$Z = \int_{\mathbb{R}^2} e^{-\frac{(x_1^2 + x_2^2)}{2}} dx_1\, dx_2 =$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \int_{\mathbb{R}} e^{-\frac{x_1^2}{2}} dx_1 \int_{\mathbb{R}} e^{-\frac{x_2^2}{2}} dx_2 = 2\pi$$

A diagonal

$$A = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$$

$$Z = \int_{\mathbb{R}} e^{-\mu_1 \frac{x_1^2}{2}} dx_1 \int_{\mathbb{R}} e^{-\mu_2 \frac{x_2^2}{2}} dx_2 =$$

$$\int_{\mathbb{R}} e^{-\mu \frac{x^2}{2}} dx = \frac{1}{\sqrt{\mu}} \int e^{-\frac{y^2}{2}} dy =$$

$$y = \sqrt{\mu}\, x$$

$$= \sqrt{\frac{2\pi}{\mu}}$$

$$Z = \frac{2\pi}{\sqrt{\mu_1 \mu_2}} = \frac{2\pi}{\sqrt{\det A}}$$

General A.

Q 2×2 such that

$$Q A Q^T = D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

$$Q Q^T = Id$$

$$\det Q Q^T = \det Q \det Q^T =$$

$$= (\det Q)^2 = \det Id = 1$$

$$\det Q = \pm 1$$

$$Z = \iint_{\mathbb{R}^2} e^{-(\vec{x}, A\vec{x})} dx_1 dx_2$$

$$\vec{y} = Q\vec{x} \qquad \vec{x} = Q^{-1}\vec{y} = Q^T \vec{y}$$

$$(\vec{x}, A\vec{x}) = (Q^T\vec{y}, A Q^T \vec{y}) =$$

$$= (\vec{y}, Q A Q^T \vec{y}) = (\vec{y}, D \vec{y})$$

$$Z = \iint_{\mathbb{R}^2} e^{-(\vec{y}, D\vec{y})} \, dy_1 \, dy_2 =$$

$$= \frac{2\pi}{\sqrt{d_1 d_2}} = \frac{2\pi}{\sqrt{\det A}}$$

$$f_{X_1,X_2}(x_1,x_2) = \sqrt{\frac{\det A}{2\pi}} \, e^{-(\vec{x}, A\vec{x})}$$

$$C = A^{-1}$$

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{\sqrt{2\pi \det C}} e^{-(\vec{x}, C^{-1}\vec{x})}$$

Q diagonalize A

$$\vec{y} = Q\vec{x}$$

$$f_{Y_1,Y_2}(y_1,y_2) = \sqrt{\frac{d_1}{2\pi}} e^{-\frac{d_1 y_1^2}{2}} \sqrt{\frac{d_2}{2\pi}} e^{-\frac{d_2 y_2^2}{2}}$$

Y_1 and Y_2 are independent.

Y_1 is a normal r.v. with mean 0 and variance $\frac{1}{d_2}$

Normal r.v. with mean μ and variance σ^2

$$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$N(\mu, \sigma^2)$$

$$\frac{\sqrt{\det A}}{2\pi} e^{-\frac{(\bar{x},\, A\bar{x})}{2}}$$

bivariate normal distribution

$$A = \frac{1}{\sqrt{1-\rho^2}} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

Bivariate Normal Standard

$X \quad Y \qquad g(X, Y) = Z$

$$\mathbb{E}(Z) = \iint_{\mathbb{R}^2} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

$$\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \operatorname{cov}(X, Y)$$

if $X \perp\!\!\!\perp Y \implies$

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$$

$$\operatorname{cov}(X, Y) = 0$$

$$\not\Leftarrow$$

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if X, Y are bivariate Normal

$$\operatorname{cov}(X, Y) = 0 \implies X \perp\!\!\!\perp Y$$